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A CLASS OF RIEMANNIAN HOMOGENEOUS SPACES

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In M. Berger's classification [1] of normal Riemannian homogeneous spaces of strictly positive curvature, appear various classes of Riemannian homogeneous metrics which can be put on odd-dimensional spheres — but which do not have constant curvature. In this paper, we investigate one of these classes, viz. $\{SU(n+1) \times R/SU(n) \times R\}_\alpha \equiv_{def} M_\alpha^n$ (cf. details below). In particular, we (i) calculate the conjugate locus (in the tangent space) of any point, (ii) calculate the totally geodesic submanifolds of constant curvature, and (iii) consider closed geodesics in M_α^n and their relationship to a lemma of W. Klingenberg [5, Theorem 1].

§ 1 concerns itself with the explicit construction of M_α^n, and § 2 is devoted to (i). The key tool of § 2 is the writing of Jacobi's equations of geodesic deviation in the canonical connection of G/H. (This connection is not the Levi-Civita connection, but nevertheless has the same geodesics.) For the necessary background, the reader is referred to [3]. In § 3 we dispose of (ii), in § 4 we calculate the pinching of M_α^n, and in § 5 we discuss (iii). § 5 is a direct generalization of M. Berger's argument in [2, pp. 9–12], and § 6 consists of remarks relating the spaces M_α^n to problems in Riemannian geometry.

1. The space M_α^n

We first let E_{jk} denote the matrix, whose r-th row and s-th column are given by $\delta_{jr}\delta_{ks}$, i.e., E_{jk} has a 1 in the j-th row and k-th column and zeros elsewhere, and we set

$$\begin{aligned} A_{jk} &= \sqrt{-1}\,(E_{jj} - E_{kk}) \,, \\ B_{jk} &= \qquad (E_{jk} - E_{kj}) \,, \\ C_{jk} &= \sqrt{-1}\,(E_{jk} + E_{kj}) \,. \end{aligned}$$

Then a basis of $\mathfrak{a}_n$ = Lie algebra of $SU(n+1)$ is given by $\{A_{l,l+1}: l = 1, \cdots, n; B_{rj}, C_{rj}: 1 \leq r < j \leq n+1\}$. For a bi-invariant metric on $\mathfrak{a}_n$ we choose

$$\langle X, Y \rangle = -\frac{1}{2}\,\text{trace}\, XY \,.$$

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One then calculates an orthonormal basis of $\mathfrak{a}_{n-1} \subset \mathfrak{a}_n$ (where $\mathfrak{a}_{n-1}$ is imbedded in $\mathfrak{a}_n$ in the usual manner) to be $\{S_j: j = 1, \cdots, n-1; B_{rj}, C_{rj}: 1 \leq r < j \leq n\}$ where

$$S_j = (1/\alpha_j) \sum_{l=1}^{j} l A_{l,l+1} ,$$

$$\alpha_j = \{j(j+1)/2\}^{1/2} .$$

An orthonormal basis of $\mathfrak{m}$ = orthogonal complement of $\mathfrak{a}_{n-1}$ in $\mathfrak{a}_n$ is then seen to be $\{S_n, B_{r,n+1}, C_{r,n+1}: r = 1, \cdots, n+1\}$. For completeness we list the Lie multiplication table:

$$\begin{aligned}
[A_{rj}, A_{kl}] &= 0 , \\
[A_{rj}, B_{kl}] &= \delta_{rk}C_{rl} - \delta_{rl}C_{rk} - \delta_{jk}C_{jl} + \delta_{jl}C_{jk} , \\
[A_{rj}, C_{kl}] &= -\delta_{rk}B_{rl} - \delta_{rl}B_{rk} + \delta_{jk}B_{jl} + \delta_{jl}B_{jk} , \\
[B_{rj}, B_{kl}] &= \delta_{jk}B_{rl} - \delta_{jl}B_{rk} - \delta_{rk}B_{jl} + \delta_{rl}B_{jk} , \\
[B_{rj}, C_{kl}] &= \delta_{jl}C_{rk} + \delta_{jk}C_{rl} - \delta_{rl}C_{jk} - \delta_{rk}C_{jl} , \\
[C_{rj}, C_{kl}] &= -\delta_{jk}B_{rl} - \delta_{jl}B_{rk} - \delta_{rk}B_{jl} - \delta_{rl}B_{jk} .
\end{aligned}$$

As usual, $[\mathfrak{a}_{n-1}, \mathfrak{m}] \subset \mathfrak{m}$. Furthermore, by direct calculation one obtains

$$(1) \qquad [\mathfrak{a}_{n-1}, S_n] = 0 ,$$

$$(2) \qquad \dim [\mathfrak{a}_{n-1}, B_{1,n+1}] = 2n - 1 .$$

One therefore obtains: *$Ad(SU(n))$ acts transitively on the unit sphere in* $\mathfrak{m}$ *spanned by* $\{B_{j,n+1}, C_{j,n+1}; j = 1, \cdots, n\}$.

We now consider the direct orthogonal sum of $\mathfrak{a}_n \oplus \boldsymbol{R} = \mathfrak{g}_n$, $\boldsymbol{R}$ = real numbers, with D a basis element of $\boldsymbol{R}$ of length 1, $[\mathfrak{a}_n, \boldsymbol{R}] = 0$, and set

$$\mathfrak{h}_\alpha = \text{linear span } \{S_1, \cdots, S_{n-1}, \cos\alpha \cdot S_n + \sin\alpha \cdot D, B_{rj}, C_{rj}: 1 \leq r < j \leq n\} ,$$

$$\mathfrak{m}_\alpha = \text{linear span } \{\sin\alpha \cdot S_n - \cos\alpha \cdot D, B_{j,n+1}, C_{j,n+1}: 1 \leq j \leq n\} ,$$

$0 < \alpha \leq \pi/2$, $G_n = \exp \mathfrak{g}_n$, $H_\alpha = \exp \mathfrak{h}_\alpha$, and $M_\alpha^n = G_n/H_\alpha$, where exp denotes the exponential map of the Lie algebra to the Lie group it generates. $\pi: G_n \to M_\alpha^n$ denotes the canonical projection, and $d\pi$, the induced linear map, identifies $\mathfrak{m}_\alpha$ with the tangent space of M_α^n at $o = \pi(H_\alpha)$. A Riemannian metric on M_α^n is obtained by restricting the metric on $\mathfrak{g}_n$ to $\mathfrak{m}_\alpha \times \mathfrak{m}_\alpha$ and then translating with G_n. Of course M_α^n is now Riemannian homogeneous; also M_α^n is topologically a sphere. From (1, 2) we have

Proposition 1. *$Ad(H_\alpha)$ is transitive on the unit tangent sphere of the orthogonal complement of* $\sin\alpha \cdot S_n - \cos\alpha \cdot D$ *in* $\mathfrak{m}_\alpha$.

2. The conjugate locus of M_α^n

Of course, by the homogeneity of M_α^n, it suffices to consider the conjugate locus of $o = \pi(H_\alpha)$. An immediate consequence of Proposition 1 is

Proposition 2. *The conjugate locus of $o = \pi(H_\alpha)$ in $\mathfrak{m}_\alpha$ is a hypersurface of revolution about the line generated by* $\sin\alpha\cdot S_n - \cos\alpha\cdot D$.

We set

$$e_0 = \sin\alpha\cdot S_n - \cos\alpha\cdot D\ ,$$
$$e_{2j-1} = B_{j,n+1}\ ,$$
$$e_{2j} = C_{j,n+1}\ , \qquad j = 1, \cdots, n\ .$$

Then by Proposition 2, the complete conjugate locus is known by finding the conjugate points of o along the geodesics emanating from o with initial unit velocity vector:

$$\xi_\theta = e_0\cos\theta + e_1\sin\theta\ , \qquad \theta\in[-\pi, \pi]\ .$$

ξ_θ is completed to an orthonormal basis of $\mathfrak{m}_\alpha$, viz., $\mathfrak{m}_\alpha$ = linear span $\{\xi_\theta, \zeta_\theta = -e_0\sin\theta + e_1\cos\theta, e_2, \cdots, e_{2n}\}$.

Let $\varepsilon_\theta(t)$ denote the geodesic satisfying $\varepsilon_\theta(0) = o$, $\varepsilon_\theta'(0) = \xi_\theta$. To solve Jacobi's equations along $\varepsilon_\theta(t)$, we write them in the canonical connection (of the second kind)—cf [3].

The torsion T and curvature B, tensors of the canonical connection, are given at 0 by

$$T(X, Y) = [X, Y]_{\mathfrak{m}_\alpha}\ , \qquad B(X, Y)Z = [[X, Y]_{\mathfrak{h}_\alpha}, Z]\ .$$

For future reference we list the torsion and curvature relative to the basis $e_0, \cdots, e_{2n}$. First we set

$$\beta = (n+1)(\sin\alpha)\alpha_n = \{2(n+1)/n\}^{1/2}\sin\alpha\ . \tag{3}$$

Then

$$T(e_0, e_{2k-1}) = \beta e_{2k}\ , \tag{4}$$

$$T(e_0, e_{2k}) = -\beta e_{2k-1}\ , \tag{5}$$

$$T(e_{2k-1}, e_{2k}) = \beta e_0\ . \tag{6}$$

Otherwise, $T(e_j, e_k) = 0$, $B(e_0, e_j)\cdot e_k = B(e_j, e_k)\cdot e_0 = 0$ for all $j, k = 0, 1, \cdots, 2n$. Henceforth, assume $j \neq k$. Then

$$B(e_{2j-1}, e_{2k-1})\cdot \begin{cases} e_{2s-1} = -\delta_{ks}e_{2j-1} + \delta_{js}e_{2k-1}\ , \\ e_{2s} \quad = \delta_{ks}e_{2j} - \delta_{js}e_{2k}\ , \end{cases} \tag{7}$$

$$(8)\qquad B(e_{2j-1}, e_{2j}\ )\cdot \begin{cases} e_{2s-1} = 2\delta_{js}e_{2j} + (2-\beta^2)e_{2s}\ , \\ e_{2s} \ \ = -2\delta_{js}e_{2j-1} - (2-\beta^2)e_{2s-1}\ , \end{cases}$$

$$(9)\qquad B(e_{2j-1}, e_{2k}\ )\cdot \begin{cases} e_{2s-1} = \delta_{ks}e_{2j} + \delta_{js}e_{2k}\ , \\ e_{2s} \ \ = -\delta_{ks}e_{2j-1} - \delta_{js}e_{2k-1}\ , \end{cases}$$

$$(10)\qquad B(e_{2j}\ , e_{2k}\ )\cdot \begin{cases} e_{2s-1} = -\delta_{ks}e_{2j-1} + \delta_{js}e_{2k-1}\ , \\ e_{2s} \ \ = -\delta_{ks}e_{2j} \ \ + \delta_{js}e_{2k}\ . \end{cases}$$

Now let T_θ of B_θ be the linear transformations given by

$$(11)\qquad T_\theta\cdot X = T(\xi_\theta, X)\ ,$$

$$(12)\qquad B_\theta\cdot X = B(\xi_\theta, X)\xi_\theta\ .$$

Then relative to the basis $\zeta_\theta, e_2, \cdots, e_{2n}$, we have

$$(13)\qquad T_\theta\cdot\zeta_\theta = \beta e_2\ , \qquad B_\theta\cdot\zeta_\theta = 0\ ,$$

$$(14)\qquad T_\theta\cdot e_2 = -\beta\zeta_\theta\ , \qquad B_\theta\cdot e_2 = (4-\beta^2)(\sin^2\theta)e_2\ ,$$

$$(15)\qquad T_\theta\cdot e_{2k-1} = \beta(\cos\theta)e_{2k}\ , \qquad B_\theta\cdot e_{2k-1} = (\sin^2\theta)e_{2k-1}\ ,$$

$$(16)\qquad T_\theta\cdot e_{2k} = -\beta(\cos\theta)e_{2k-1}\ , \qquad B_\theta\cdot e_{2k} = (\sin^2\theta)e_{2k}\ ,$$

where $k \geq 2$. In particular, T_θ and B_θ *have the same invariant subspaces for all* θ.

As in [3], let $a_i(t)$, $i = 1, \cdots, 2n$, be a parallel (with respect to the canonical connection) orthonormal frame along $\varepsilon_\theta(t)$ for which $a_1(0) = \zeta_\theta$, $a_i(0) = e_i$, $i = 2, \cdots, 2n$. If we write for any vector field $\eta(t)$ along ε_θ, $\eta(t) = \sum_{i=1}^{2n} \eta_i(t)a_i(t)$, then Jacobi's equations along ε_θ read as:

$$(17)\qquad \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}'' + \begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}' + \begin{pmatrix}0 & 0\\ 0 & (4-\beta^2)\sin^2\theta\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} = 0\ ,$$

$$(18)\qquad \begin{pmatrix}\eta_{2k-1}\\ \eta_{2k}\end{pmatrix}'' + \begin{pmatrix}0 & -\beta\cos\theta\\ \beta\cos\theta & 0\end{pmatrix}\begin{pmatrix}\eta_{2k-1}\\ \eta_{2k}\end{pmatrix} + \begin{pmatrix}\sin^2\theta & 0\\ 0 & \sin^2\theta\end{pmatrix}\begin{pmatrix}\eta_{2k-1}\\ \eta_{2k}\end{pmatrix} = 0\ ,$$

$k = 2, \cdots n$. Thus one obtains a basis of solutions vanishing at $t = 0$, $\Lambda_1(t), \cdots, \Lambda_{2n}(t)$ such that

$$(19)\qquad \langle \Lambda_{2k-i}(t), a_j(t)\rangle = 0\ ,$$

for all t, where $i = 0, 1$; $k = 1, \cdots, n$; $j = 1, \cdots, 2k-2, 2k+1, \cdots, 2n$, and

$$(20)\qquad \frac{D}{dt}\Lambda_i|_{t=0} = k_i a_i(0)\ ,$$

$i = 1, \cdots, 2n$, $k_i =$ constant, where D/dt denotes covariant differentiation in the canonical connection along $\varepsilon_\theta(t)$. Using the Remark of [4] and (20) one sees that $\Lambda_2, \cdots, \Lambda_{2n}$ form a basis of isotropic Jacobi fields along ε_θ, $\varepsilon'_\theta(0) \neq \pm e_0$; otherwise, no Jacobi fields are isotropic.

We turn to the solving of (17). Set

$$\gamma = (4 - \beta^2) \sin^2 \theta \,, \tag{21}$$

$$\sigma = (4 \sin^2 \theta + \beta^2 \cos^2 \theta)^{1/2} \,. \tag{22}$$

Then the eigenvalues λ of (17) are easily seen to be $\lambda^2 = 0$ and $\lambda = \pm\sqrt{-1}\,\sigma$. Standard calculation yields

$$\Lambda_1(t) = \{-(\gamma/\beta)t - (\beta/\sigma) \sin \sigma t\} a_1(t) + \{1 - \cos \sigma t\} a_2(t) \,, \tag{23}$$

$$\Lambda_2(t) = \{(\beta/\sigma)(1 - \cos \sigma t)\} a_1(t) + (\sin \sigma t) a_2(t) \,. \tag{24}$$

Conjugate points obtained from linear combinations of Λ_1 and Λ_2 have path values t for which

$$0 = f(t) = (2\beta/\sigma)(1 - \cos \sigma t) + (\gamma/\beta) t \sin \sigma t \,. \tag{25}$$

Also,

$$f'(t) = \{(2\beta^2 + \gamma)/\beta\} \sin \sigma t + (\gamma\sigma/\beta) t \sin \sigma t \,.$$

Now $f(2\pi k/\sigma) = 0$ for all integers k. Indeed these values are precisely the zeros of $\Lambda_2(t)$. However, for sufficiently small $|t| > 0$, $f(t)$ is given by $f(t) = \{\beta\sigma + (\gamma\sigma/\beta)\} t^2 + \cdots > 0$; and $f'(2\pi k/\sigma) > 0$ if and only if $\gamma(\theta) \neq 0$, i.e., $\xi_\theta \neq \pm e_0$. Thus, for $\xi_\theta \neq \pm e_0$ and every integer k, a linear combination of Λ_1 and Λ_2 vanishes for some $t_0 \in (2\pi k/\sigma, 2\pi(k+1)/\sigma)$. Note that the Jacobi field in question is nonisotropic.

We now turn to (18). Since the matrices

$$\mathscr{T} = \begin{pmatrix} 0 & -\beta \cos \theta \\ \beta \cos \theta & 0 \end{pmatrix}, \qquad \mathscr{B} = \begin{pmatrix} \sin^2 \theta & 0 \\ 0 & \sin^2 \theta \end{pmatrix}$$

commute, the system can be changed in the usual manner by letting

$$\begin{pmatrix} b_{2k-1}(t) \\ b_{2k}(t) \end{pmatrix} = \exp\left(\frac{t}{2}\mathscr{T}\right) \cdot \begin{pmatrix} a_{2k-1}(t) \\ a_{2k}(t) \end{pmatrix},$$

and $\eta = \hat{\eta}_{2k-1} b_{2k-1} + \hat{\eta}_{2k} b_{2k}$. The result is then

$$\hat{\eta}''_{2k-1} + (\sigma^2/4)\hat{\eta}_{2k-1} = 0 \,,$$
$$\hat{\eta}''_{2k} + (\sigma^2/4)\hat{\eta}_{2k} = 0 \,,$$

and therefore $\Lambda_{2k-1}(t) = \sin \sigma t/2 \cdot b_{2k-1}(t)$, $\Lambda_{2k}(t) = \sin \sigma t/2 \cdot b_{2k}(t)$. Geometrically, $b_{2k-1}(t)$, $b_{2k}(t)$ are Riemannian-parallel orthonormal vector fields with initial values e_{2k-1}, e_{2k} respectively, and $\sigma^2/4$ is the sectional curvature of the 2-sections $(\varepsilon'_\theta(t), b_{2k-1}(t))$ and $(\varepsilon'_\theta(t), b_{2k}(t))$ for all t; cf. [3, p. 244]. The zeros of $\Lambda_3, \cdots, \Lambda_{2n}$ have arc values $t = 2\pi k/\sigma$. We summarize our results in

Theorem 1. *Let* $\varepsilon: (-\infty, \infty) \to M^n_\alpha$. $\|\varepsilon'\| = 1$, $\varepsilon(0) = o$ *be a geodesic in* M^n_α, θ *the angle from* e_0 *to* $\varepsilon'(0)$, *and* β *and* σ *as given in* (3) *and* (22) *respectively. Then all isotropic Jacobi fields along* ε *vanish for* $t = 2\pi k/\sigma$, $k = 0, \pm 1, \pm 2, \cdots$, *and there exists a non-isotropic Jocobi field along* ε *vanishing for* $t = 0$ *and some* $t_0 < 2\pi/\sigma$ *whenever* $\varepsilon'(0) \neq \pm e_0$. *For* $\varepsilon'(0) = \pm e_0$, *no Jacobi fields along* ε *are isotropic, and all Jacobi fields vanish for* $t = 2\pi k/\sigma = 2\pi k/\beta$. *The first conjugate locus of o therefore consists entirely of non-isotropic conjugate points, and the generating set of the conjugate locus, in* $\mathfrak{m}_\alpha$, *in the* (e_0, e_1)*-plane is given by the solutions of* (25) *for each* θ, *where* $\gamma(\theta), \sigma(\theta)$ *are given by* (21), (22). *Finally, the generating set is symmetric with respect to the* e_0*-axis and* e_1*-axis.*

3. Totally geodesic submanifolds of constant curvature

Let $\mathfrak{m}^0$ be a subspace of $\mathfrak{m}_\alpha$. Then Theorem 2 of [7] states that the subset $\operatorname{Exp} \mathfrak{m}^0$ (where Exp is the Riemannian exponential map) is a totally geodesic submanifold of G/H if for every $X, Y, Z \in \mathfrak{m}^0$, $T(X, Y)$ and $B(X, Y)Z \in \mathfrak{m}^0$. (This is not A. Sagle's original statement of the theorem. To pass from his formulation to ours, one uses [3, (3)].) We remark that $\operatorname{Exp} \mathfrak{m}^0$ is homogeneous by (i) Sagle's explicit construction and (ii) an unpublished result of S. Kobayashi that every totally geodesic submanifold of a homogeneous space is homogeneous.

For each $k = 1, \cdots, n$, let V_k be the subspace of $\mathfrak{m}_\alpha$ generated by e_{2k-1}, e_{2k}.

Theorem 2. *Let* $\mathfrak{m}^0$ *be any subspace of* $\mathfrak{m}_\alpha$ *generated by unit vectors* $X_1, \cdots, X_k$ *where* $X_k \in V_k$, $k = 1, \cdots, n$. *Then* $\operatorname{Exp} m^0$ *is a totally geodesic submanifold, of constant curvature* 1, *of* M^n_α. *Furthermore, n is the maximal dimension for the total geodesy of a submanifold of* M^n_α *of constant curvature* 1 *for all* α *when* $n > 1$, *and for* $\alpha < \pi/2$ *where* $n = 1$. *Finally, the subspace* $\mathfrak{m}^0$ *described is the only subspace of* $\mathfrak{m}_\alpha$ *generating a totally geodesic submanifold of constant curvature* 1 *through o.*

Proof. First, for any $X, Y \in \mathfrak{m}^0$, $T(X, Y) = 0$; and $X, Y, Z \in \mathfrak{m}^0$ implies $B(X, Y)Z \in \mathfrak{m}^0$ by checking (7)–(10). Also, since T restricted to $\mathfrak{m}^0 \times \mathfrak{m}^0$ vanishes identically, we have that $B(X, Y)Z = R(X, Y)Z$ for all $X, Y, Z \in \mathfrak{m}^0$, where $R(X, Y)Z$ denotes the Riemannian curvature tensor; cf. [3, (3)]. One checks that the Riemannian sectional curvature is 1. To increase the dimension of $\mathfrak{m}^0$ would either (i) yield a non-trivial projection of $\mathfrak{m}^0$ onto e_0, which would contradict constant curvature assumption (if not also total geodesy), or (ii), for some $k = 1, \cdots, n$, yield a projection of $\mathfrak{m}^0$ onto all of V_k which would contradict total geodesy by (6). The last statement in the theorem also follows from (i) and (ii).

4. Riemannian curvature and the pinching of M_α^n

R denotes the Riemannian curvature tensor of M_α^n, R_θ the linear transformation $R_\theta \cdot Y = R(\xi_\theta, Y)\xi_\theta$. Then by (14) of [3], we have

$$R_\theta = B_\theta - \frac{1}{4}(T_\theta)^2 ,$$

which implies

$$\begin{aligned} R_\theta \cdot \zeta_\theta &= (\beta^2/4) \cdot \zeta_\theta , \\ R_\theta \cdot e_2 &= \{\beta^2/4 + (4 - \beta^2) \sin^2 \theta\} e_2 , \\ R_\theta \cdot e_{2k-1} &= \{\beta^2/4 \cos^2 \theta + \sin^2 \theta\} e_{2k-1} , \\ R_\theta \cdot e_{2k} &= \{\beta^2/4 \cos^2 \theta + \sin^2 \theta\} e_{2k} , \qquad k = 2, \cdots, n . \end{aligned}$$

Therefore the set $\operatorname{curv}(M_\alpha^n)$ of real numbers assumed as values of sectional curvatures is given by

(26) $$\operatorname{curv}(M_\alpha^n) = [(n+1)\sin^2\alpha/2n,\ 4 - (3(n+1)\sin^2\alpha/2n)] ,$$

and the pinching δ_α^n is given by

(27) $$\delta_\alpha^n = \frac{(n+1)\sin^2\alpha}{8n - 3(n+1)\sin^2\alpha} .$$

5. Closed geodesics and Klingenberg's lemma

We first note that by Theorem 2, all geodesics emanating from o with initial velocity vector in the orthogonal complement of e_0 in $\mathfrak{m}_\alpha$ are simply closed and have length 2π.

We now note that $\exp(2\pi\alpha_n/n)S_n \in SU(n)$, and that $2\pi\alpha_n/n$ is the first value of t for which $\exp tS_n \in SU(n)$. Also, recall that $[\mathfrak{h}_\alpha, S_n] = 0$. Thus the group generated by $\mathfrak{h}_\alpha \oplus S_n = \mathfrak{a}_{n-1} \oplus \boldsymbol{R} \oplus \boldsymbol{R}$ is a cylinder with generator $SU(n) \times \boldsymbol{R}$ and base circle of length $2\pi\alpha_n/n$. Now geodesics in G_n/H_α through o are projections of the one parameter subgroups of G_n generated by the elements of $\mathfrak{m}_\alpha$, and it is easy to see that $\gamma(t) = \pi(\exp t\, e_0)$ is a closed (and hence simply closed [6, Th. 3]) geodesic of length $(2\pi\alpha_n/n)\sin\alpha$. Since the maximum curvature of M_α^n is given by $4 - (3(n+1)/2n)\sin^2\alpha$, Klingenberg's lemma [6, Th. 1] for odd dimensions would imply

$$(2\pi\alpha_n/n)\sin\alpha \geq 2\pi/\{4 - (3(n+1)/2n)\sin^2\alpha\}^{1/2} ,$$

from which one implies $\sin^2\alpha \geq 2n/(3n+3)$. Thus for $\sin^2\alpha < 2n/(3n+3)$, Klingenberg's lemma is false. For $\sin^2\alpha = 2n/(3n+3)$ the pinching of M_α^n is $1/9$ *for all* n. One wonders$\cdots$.

6. Remarks

(A) For any normal Riemannian homogeneous space G/H with orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and for any $X \in m$, we let $T_X = T(X, \ )$ and $B_X = B(X, \ )X$. We define G/H to be *quasi-symmetric* if for all $X \in m$, T_X and B_X commute. As noted in [3], T_X is skew-symmetric and B_X is symmetric. In a quasi-symmetric space, Jacobi's equations split into subsystems of 2 by 2's which are easily handled. Indeed, M_α^n is quasi-symmetric, and it is easy to see that for dimension 3 all G/H are quasi-symmetric. Are all homogeneous G/H quasi-symmetric?

(B) In [3] we proved that if every conjugate point of a simply connected normal Riemannian G/H of rank one is isotorpic, then G/H is homeomorphic to a symmetric space of rank one. For the spaces M_α^n, one might say that the $Ad(H_\alpha)$ acts *almost transitively* on m_α. Yet, by Theorem 1, all the points of the first conjugate locus are non-isotropic. Also, if the linear isotropy is transitive on unit tangent spheres, the space is Riemannian symmetric (this is only known heretofore by classification arguments). We therefore

Conjecture. *If every conjugate point of a simply connected normal Riemannian G/H of rank one is isotropic, then G/H is isometric to a Riemannian symmetric space of rank one*

Added in proof. We note that A. Sagle's condition for total geodesy of submanifolds of a reductive Riemannian homogeneous space G/H is only sufficient but not necessary. More preceisely, Sagle's theorem says what Kobayashi's does not, viz., if $T(X, Y)$, $B(X, Y)Z \in \mathfrak{m}^0$ for all $X, Y, Z \in \mathfrak{m}^0$, then Exp $\mathfrak{m}^0$ is homogeneous *relative to a subgroup of* G. Inspection as in the proof of Theorem 2 shows that we have indeed ennumerated *all* totally geodesic submanifolds of G/H through $\pi(H)$.

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